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BOUNDARY VALUE PROBLEMS FOR EQUATION $\varepsilon y'' = f(x, y, y')$ FOR SMALL VALUES OF ε

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BOUNDARY VALUE PROBLEMS FOR EQUATION $\varepsilon y'' = f(x, y, y')$ FOR SMALL VALUES OF ε /429*

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ABSTRACT. The behavior of solutions of several boundary value problems for the equation $\varepsilon y'' = f(x, y, y')$ when ε tends toward zero is studied. Theorems 1-6 are formulated, and the proof of these theorems is presented.

In this report we will study the behavior of the solutions of several boundary value problems for the equation

$$\varepsilon y'' = f(x, y, y') \quad (\varepsilon > 0) \tag{1}$$

when ε tends toward zero. In the case where $f(x, y, y') \equiv A(x, y,)y' + B(x, y)$, the first boundary value problem -- i.e., the problem with the boundary conditions $y(a) = y_0$, $y(b) = y_1$ -- is examined in (Refs. 1, 2). Oleynik and Zhizhin (Ref. 3) have also studied the first boundary value problem in the case in which $f(x, y, y') \equiv A(x, y)y' + F(x, y, y')$, where F is a bounded function.

For the solutions $y_\varepsilon(x)$ to equation (1) in the interval [a, b] which satisfy boundary conditions

$$y_\varepsilon(a) = y_\varepsilon(b) = 0, \tag{2}$$

the following theorems are valid:

Theorem 1. *If the following conditions are fulfilled: (1) in interval $a \leq x \leq b, |y| < d$, equation (1) has a solution $y_\varepsilon(x)$ satisfying conditions (2) for every sufficiently small $\varepsilon > 0$; (2) the derivative $y'_\varepsilon(x)$ for every stipulated ε is a bounded function of x in $[a, b]$; (3) function f is continuous together with derivatives f_x, f_y, and $f_{y'}$ in region G: $a \leqslant x \leqslant b$, $|y| \leqslant d$, $|y'| < \infty$; (4) $f_y \leq -k < 0$ on G; then on $[a, b]$ solution $u(x)$ exists to equation*

$$f(x, u, u') = 0, \tag{3}$$

which satisfies condition $u(b) = 0$, and equations

$$|y_\varepsilon(x) - u(x)| \leqslant |u(a)| e^{C_1(x-a)} e^{-\frac{k}{\varepsilon}(x-a)} + C_2\varepsilon, \tag{4}$$

$$|y'_\varepsilon(x) - u'(x)| \leqslant \frac{C'_1}{\varepsilon} e^{-\frac{k}{\varepsilon}(x-a)} + C'_2\varepsilon, \quad a + \varepsilon \leqslant x \leqslant b, \tag{5}$$

hold true, where C_1, C_2, C'_1, and C'_2 are constants not depending on ε.

Let us note that conditions (1) and (2) of Theorem 1 will always be satisfied if $f_y(x, y, 0) > 0$ and $|f(x, y, y')| \leqslant K(1 + y'^2)$, where

* Numbers in the margin indicate pagination in the original foreign text.

K is constant (Ref. 4).

Theorem 2. *If the following conditions are fulfilled: (1) solution $u(x)$ to equation (3), on condition that $u(b) = 0$, exists in $[a, b]$; (2) function f is continuous together with derivatives f_x, f_y, and $f_{y'}$, in region G: $a \leqslant x \leqslant b$, $|y-u(x)| \leqslant d$, $|y'| < \infty$; $(d > 0, u(a) - d < 0 < u(a) + d)$; (3) $f_{y'} \leqslant -k < 0$ in G: (4) $|f(x, y, y')| \leqslant \chi(|y'|)$ in G, where $\chi(z)$ is a continuous positive function when $0 \leqq z < \infty$ and such that $\int_0^\infty \frac{z\,dz}{\chi(z)} = \infty$, then for every sufficiently small $\varepsilon > 0$ equation (1) has a solution $y_\varepsilon(x)$ which satisfies conditions (2), and inequalities (4) and (5) hold true.* /430

Proof. Functions

$$\bar{\omega}(x) = \begin{cases} u(x) + \frac{\varepsilon M\gamma}{l}(e^{\lambda_2(x-b)} - 1), & u(a) \geqslant 0; \\ u(x) - u(a)e^{\lambda_1(x-a)} + \frac{\varepsilon M\gamma}{l}(e^{\lambda_2(x-b)} - 1), & u(a) < 0; \end{cases}$$

$$\underline{\omega}(x) = \begin{cases} u(x) - u(a)e^{\lambda_1(x-a)} - \frac{\varepsilon M\gamma}{l}(e^{\lambda_2(x-b)} - 1), & u(a) \geqslant 0; \\ u(x) - \frac{\varepsilon M\gamma}{l}(e^{\lambda_2(x-b)} - 1), & u(a) < 0, \end{cases}$$

where γ is any number > 1; $\lambda_1 = \frac{-k - \sqrt{k^2 - 4\varepsilon l}}{2\varepsilon}$, $\lambda_2 = \frac{-k + \sqrt{k^2 - 4\varepsilon l}}{2\varepsilon}$;

$|f_y(x, y, u'(x))| \leqq l$, $|u''(x)| \leqq M$, satisfy inequalities $\varepsilon\bar{\omega}''(x) < f(x, \bar{\omega}(x), \bar{\omega}'(x))$, $\varepsilon\underline{\omega}''(x) > f(x, \underline{\omega}(x)\ \underline{\omega}'(x))$ on $[a, b]$ for all sufficiently small values of $\varepsilon > 0$. In addition, $\bar{\omega}(a) > 0, \bar{\omega}(b) \geqslant 0, \underline{\omega}(a) < 0, \underline{\omega}(b) \leqslant 0$, and $\underline{\omega}(x) < \bar{\omega}(x)$ when $a < x < b$.

Hence it follows by virtue of the theorem advanced by Bernshteyn (Ref. 5) that solution $y_\varepsilon(x)$ exists, and inequality $\underline{\omega}(x) \leqslant y_\varepsilon(x) \leqslant \bar{\omega}(x)$ holds. From this inequality, we derive inequality (4). Inequality (5) follows from Theorem 1.

Let us note that Theorem 2, as follows from Bernshteyn (Ref. 4), may prove to be false if condition (4) is not fulfilled.

Let us examine several cases where condition (3) of Theorem 2 is not fulfilled. Let us assume that equation (1) has the form

$$\varepsilon y'' = \Psi_1(x, y) \quad (\varepsilon > 0). \tag{1'}$$

Theorem 3. *If the following conditions are fulfilled: 1) there is a function $\phi(x)$ in $[a, b]$ such that $\Psi(x, \phi(x)) = 0$; (2) function Ψ is continuous together with derivative Ψ_y in region B: $a \leqslant x \leqslant b$, $\alpha_1(x) \leqslant y - \varphi(x) \leqslant \alpha_2(x)$, where $\alpha_1(x)$ and $\alpha_2(x)$ are functions, continuous in $[a, b]$; $(\alpha_1(x) < 0 < \alpha_2(x)$, $\varphi(a) + \alpha_1(a) < 0 < \varphi(a) + \alpha_2(a)$, $\varphi(b) + \alpha_1(b) < 0 < \varphi(b) + \alpha_2(b))$; (3) $\Psi_y \geqq m > 0$ in*

B, then for every sufficiently small value of $\varepsilon > 0$ equation (1') has in B the unique solution $y_\varepsilon(x)$ which satisfies conditions (2). The solutions $y_\varepsilon(x)$ converge uniformly in $[a + \delta, b - \delta]$ $(\delta > 0)$ to $\phi(x)$ when $\varepsilon > 0$, and ε strives toward zero. If, in addition, the function $\phi(x)$ is twice continuously differentiable, then

$$|y_\varepsilon(x) - \varphi(x)| \leqslant |\varphi(a)| e^{-\sqrt{\frac{m}{\varepsilon}}(x-a)} + |\varphi(b)| e^{-\sqrt{\frac{m}{\varepsilon}}(b-x)} + \frac{M\varepsilon}{m},$$

where $|\phi''(x)| \leq M$.

In the case where $f_{y}' \neq 0$, the following theorem is true:

Theorem 4. *Let us assume that the following conditions are fulfilled: (1) some continuously differentiable solution $u(x)$ to equation (3) exists in $[a, b]$; (2) function f is continuous together with derivatives f_y and $f_{y'}$ in region G: $a \leqslant x \leqslant b$, $|y - u(x)| \leqslant d$, $|y'| < \infty$ $(d > 0, u(a) - d < 0 < u(a) + d,$ $u(b) - d < 0 < u(b) + d)$; (3) there are numbers $\delta_1 > 0, \delta_2 > 0$ $(\delta_1 + \delta_2 < b - a)$, $k_1 > 0$ and $k_2 > 0$ such that $f_{y'} \leq -k_1$ when $a \leq x \leq + \delta_1$ and $(x, y, y') \in G$, if* /431 *$u(a) \neq 0$; $f_{y'} \geq k_2$ when $b - \delta_2 \leq x \leq b$ and $(x, y, y') \in G$, if $u(b) \neq 0$; 4) $f_y(x, u(x), u'(x)) > 0$ when $\alpha + \delta_1 \leq x \leq b - \delta_2$; (5) condition (4) of Theorem 2 is fulfilled in G. Then for every sufficiently small value of $\varepsilon > 0$ equation (1) has a solution $y_\varepsilon(x)$ which satisfies conditions (2) such that $y_\varepsilon(x) \to u(x)$ when $\varepsilon \to 0$ uniformly in any interval obtained from $[a, b]$, excluding any neighborhoods near its ends where $u(x)$ does not become zero. If moreover $u(x)$ is a function which is twice continuously differentiable, then*

$$y_\varepsilon(x) - u(x)| \leqslant |u(a)| e^{C_1(x-a)} e^{-\frac{k_1}{\varepsilon}(x-a)} + |u(b)| e^{C_2(b-x)} A e^{-\frac{k_2}{\varepsilon}(b-x)} + C_3\varepsilon,$$

where C_1, C_2, and C_3 are constants not depending on ε.

The proofs of Theorems 3 and 4 are similar to that of Theorem 2.

Let us examine on [a, b] the solutions $y_\varepsilon(x)$ to equation (1) which satisfy conditions

$$\begin{aligned} &\alpha y_\varepsilon(a) + \alpha' y_\varepsilon'(a) = 0 \quad (|\alpha| + |\alpha'| > 0); \\ &\beta y_\varepsilon(b) + \beta' y_\varepsilon'(b) = 0 \quad (|\beta| + |\beta'| > 0). \end{aligned} \tag{6}$$

In the case when $\alpha' \neq 0$, the following theorem is true:

Theorem 5. *Let us assume that equation (3) has a solution $u(x)$ on $[a, b]$ which satisfies condition $\beta u(b) + \beta' u'(b) = 0$. For every such solution $u(x)$, let us assume that the following conditions are fulfilled: (1) function f is continuous together with derivatives f_x, f_y, and $f_{y'}$ in region D:*

$$a \leqslant x \leqslant b, \ |y - u(x)| \leqslant d, \ |y' - u'(x)| \leqslant \left(\frac{|\alpha u(a) + \alpha' u'(a)|}{|\alpha'|} + h\right) e^{-C(x-a)} + r(x),$$

where $\alpha' \neq 0$, *d, h, and C are positive constants and r(x) is a positive continuous function on* $[a, b]$; *(2)* $f_{y'} \leq -k < 0$ *in D; (3)* $\beta' f_y(b, u(b), u'(b)) - \beta f_{y'}(b, u(b), u'(b)) \neq 0$. *Then for every sufficiently small value of* $\varepsilon > 0$ *and for every u(x) equation (1) has a unique solution* $y_\varepsilon(x)$ *which satisfies conditions (6). Moreover, inequalities*

$$|y_\varepsilon(x) - u(x)| \leqslant C_1\varepsilon,$$

$$|y'_\varepsilon(x) - u'(x)| \leqslant \frac{|\alpha u(a) + \alpha' u'(a)|}{|\alpha'|} e^{-\frac{k}{\varepsilon}(x-a)} + C_2\varepsilon, \tag{7}$$

hold where C_1 *and* C_2 *are constants independent of* ε.

Proof. Constants $d = d^*$, $h = h^*$, $C = C^*$ and function $r(x) = r^*(x)$ may be fixed such that in region $a \leq x \leq b$, $|y - u(x)| \leq d^*$ there is only one solution $u(x)$ and

$$\beta' f_y(b, y, y') - \beta f_{y'}(b, y, y') \neq 0, \tag{8}$$

if point $(b, y, y') \in D^*$. Let $y_\varepsilon(x, \mu)$ be the solution to equation (1) which satisfies conditions $y_\varepsilon(a, \mu) = u(a) + \mu$, $y'_\varepsilon(a, \mu) = -\frac{\alpha}{\alpha'}[u(a) + \mu]$, , where μ is a parameter, and $u(x, \mu)$ is a solution to equation (3) on condition that $u(a, \mu) = u(a) + \mu$. Using the lemma of Nagumo (Ref. 6), we can prove that $y_\varepsilon(x, \mu)$ and /432
$y'_\varepsilon(x, \mu)$ exist in $[a, b]$ for all sufficiently small values of $|\mu|$ and $\varepsilon > 0$. Moreover, the following inequalities

$$|y_\varepsilon(x, \mu) - u(x, \mu)| < \varepsilon\left(\frac{\gamma}{k^2} + \frac{M}{l}\right) e^{\frac{l}{k}(x-a)}, \tag{9}$$

$$|y'_\varepsilon(x, \mu) - u'(x, \mu)| \leqslant \gamma e^{-\frac{k}{\varepsilon}(x-a)} + \varepsilon\left(\frac{l\gamma}{k^2} + \frac{M}{k}\right) e^{\frac{l}{k}(x-a)},$$

hold, where $|f_y| \leqslant l$, $|u''(x, \mu)| \leqslant M$, $\gamma = \frac{|\alpha u(a) + \alpha' u'(a)|}{|\alpha'|} + |\mu|\left(\frac{l}{k} + \frac{|\alpha|}{|\alpha'|}\right)$.

By virtue of relationships (8) and (9), we find $\mu = \mu^*$, $|\mu^*| < C_0\varepsilon$ (C_0 a constant not dependent on ε) such that $\beta y_\varepsilon(b, \mu^*) + \beta' y'_\varepsilon(b, \mu^*) = 0$ for all sufficiently small values of ε. Therefore $y_\varepsilon(x, \mu^*) = y_\varepsilon(x)$ is the desired solution in the region $a \leq x \leq b$, $|y - u(x)| \leq d^*$. Inequalities (7) follow from inequalities (9).

Function $w(x) = \frac{\partial y_\varepsilon(x, \mu)}{\partial \mu}$ satisfies the equation

$$\varepsilon w'' - f_{y'}(x, y_\varepsilon(x, \mu), y'_\varepsilon(x, \mu)) w' - f_y(x, y_\varepsilon(x, \mu), y'_\varepsilon(x, \mu)) w = 0$$

and conditions $w(a) = 1$, $w'(a) = -\alpha/\alpha'$. Hence, according to Kamke's congruence theorem (Ref. 7) it follows that either $\beta w(b) + \beta' w'(b) > 0$ or $\beta w(b) + \beta' w'(b) < 0$ for all sufficiently small values of ε. Consequently, the solution found above $y_\varepsilon(x)$ is unique in the region $a \leq x \leq b$, $|y - u(x)| \leq d^*$ for every sufficiently small value of ε.

If, however, $\alpha' = 0$ ($\beta' \neq 0$), then the following statement may be proved:

Theorem 6. *Let equation (3) have in* $[a, b]$ *a solution* $u(x)$ *satisfying condition* $\beta u(b) + \beta' u'(b) = 0$. *Further, let conditions (3) of Theorem 5 (2), (3) and (4) of Theorem 2 be satisfied for every such solution* $u(x)$. *Then for every sufficiently small value of* $\varepsilon > 0$ *and for every* $u(x)$, *equation (1) has a solution* $y_\varepsilon(x)$ *which satisfies condition (6)* ($\alpha' = 0$, $\beta' \neq 0$). *Here, inequalities like (4) and (5) hold.*

Let us note that the case $\varepsilon < 0$ reduces to the case under consideration if we make the substitution $x = -t$.

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